ON THE UNIQUENESS IN THE 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, we give a new regularity criterion on the uniqueness results of weak solutions for the 3D Navier-Stokes equations satisfying the energy inequality. We prove that a weak solution of the 3D Navier-Stokes equations is unique in the class of continuous solution.

1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the regularity of solutions. For the three-dimensional Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [7], it is not known if the weak solution is unique or what further assumption could make it unique. Therefore the uniqueness of weak solutions remains as an open problem. There are many results that give sufficient conditions for regularity of a weak solution [1, 2, 3, 4, 8, 10, 12, 13].

In this paper, we are interested in the problem of finding sufficient conditions for weak solutions of 3D Navier-Stokes equations such that they become regular and unique. The aim of this paper is to establish uniqueness in the class of continuous weak solutions. We prove that, if two weak solutions of the 3-dimensional Navier-Stokes are equal in such time t_0, then they are equal for all $t \geq t_0$. For the proof we use the quotient of Dirichlet to prove the uniquenesss, this quantie was used by Constantin [5] and Kukavica [6] to study the backward uniqueness in 2D Navier-Stokes equations.

2. PRELIMINARY

We denote by $H_{per}^m (\Omega)$, the Sobolev space of L-periodic functions endowed with the inner product

$$(u,v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

We define the spaces V_m as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H_{per}^m norms. V_m' denotes the dual space of V_m and V denotes the space V_0.

We denote by A the Stokes operator $Au = -\triangle u$ for $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with

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$D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact, $D(A) = V_2$. Now define the trilinear form $b(.,.,.)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{2.1}$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u, v); V \times V \to V'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \ \forall w \in V. \tag{2.2}$$

Recall that for u satisfying $\nabla.u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \tag{2.3}$$

Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, a, b, \sigma > 0, p > 1, q = \frac{p}{p-1}. \tag{2.4}$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_1, \tag{2.5}$$

where λ_1 is the smallest eigenvalue of the Stokes operator A.

3. Navier-Stokes equations

The conventional Navier-Stokes system can be written in the evolution form

$$\frac{\partial u}{\partial t} + \nu Au + B(u, u) = f, \ t > 0,$$
$$\text{div } u = 0, \text{ in } \Omega \times (0, \infty) \text{ and } u(x, 0) = u_0, \text{ in } \Omega. \tag{3.1}$$

We recall that a Leray weak solution of the Navier-Stokes equations is a solution which is bounded and weakly continuous in the space of periodic divergence-free L^2 functions, whose gradient is square-integrable in space and time and which satisfies the energy inequality. The proof of the following theorem is given in [8, 11].

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ and $f \in L^2(0, T; V_1')$, $u_0 \in V_0$ be given. Then there exists a weak solution u of (3.1) wich satisfies $u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\forall T > 0$, forthemore if $n = 2$, u is unique.*

In this paper we will be especially interested in the case where $n = 3$.

Lemma 3.2. *If u and v are two weak solutions of the 3D Navier-Stokes equations and $u - v = w \in L^\infty(0, T; V_1)$, then w is a continous function $[0, T] \to V_0$.*

Proof. We consider two solutions u and v of (3.1), and write the equation for their difference $w = u - v$. Then w satisfies

$$\begin{aligned} w_t &= -Aw - B(v, v) + B(u, u) \\ &= -Aw - B(v, w) - B(w, u). \end{aligned} \tag{3.2}$$

Clearly $Aw \in L^2(0, T; V_{-1})$, since $w \in L^2(0, T; V_1)$. So we consider, for $\phi \in V_1$ with $\|\phi\| = 1$,

$$
\begin{aligned}
|(B(v, w), \phi) - (B(w, u), \phi)| &\leq |(B(v, \phi), w) - (B(w, \phi), u)| \\
&\leq (\|u\|_{L^3} + \|v\|_{L^3}) \|\nabla \phi\|_{L^2} \|w\|_{L^6} \\
&\leq \left(\|u\|_{L^2}^{1/2} \|u\|_{L^6}^{1/2} + \|v\|_{L^2}^{1/2} \|v\|_{L^6}^{1/2} \right) \|w\|_{L^6} \\
&\leq \left(\|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2} + \|v\|_{L^2}^{1/2} \|v\|_{H^1}^{1/2} \right) \|w\|_{H^1} .
\end{aligned}
\tag{3.3}
$$

Under the assumption that $w \in L^\infty(0, T; V_1)$ it follows that $\partial_t w \in L^2(0, T; V_{-1})$. Since $w \in L^2(0, T; V_1)$ this means that $w \in C(0, T; V_0)$ [11, Lemma III. 1. 2]. □

The Lemma 3.2 gives enought regularity of w to deduce that $(\partial_t w, w) = \frac{d}{dt}\|w\|^2$. Morevore the function $w : [0, T] \to \mathbb{R}$ is bounded on compact sets of $[0, T]$.

Proposition 3.3. *If we consider $w = u - v$, the difference of two weak solutions of the 3D Navier-Stokes equations, u and v, then we have $w(t_0) = 0$ implies $w(t) = 0$ for all $t \geq t_0$.*

Proof. We obtain the equation for $w = u - v$ as

$$
\partial_t w + Aw + B(v, v) - B(u, u) = 0,
\tag{3.4}
$$

with $\text{div} w = 0$. Taking the scalar product of (3.2) with w, we have

$$
\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu\|A^{\frac{1}{2}}w\|^2 = b(w, w, u) .
\tag{3.5}
$$

Using the generalised version of Holder's inequality, we have

$$
|b(w, w, u)| \leq c_1 \|w\|_{L^4}^2 \left\|A^{\frac{1}{2}}u\right\|_{L^2} .
\tag{3.6}
$$

A straightforward application of Peetr's Theorem [8, 9]

$$
H^{(1-\theta)m}(\Omega) \subset L^{q_\theta}(\Omega), \frac{1}{q_\theta} = \frac{1}{2} - \frac{(1-\theta)m}{n},
\tag{3.7}
$$

if we consider $m = 1$ then $q_\theta \geq 4$ for $\theta \leq \frac{1}{4}$, inequality (3.6) means that

$$
|b(w, w, u)| \leq c_2 \|w\|_1^{2(1-\theta)} \|w\|_{L^2}^{2\theta} \|u\|_1.
\tag{3.8}
$$

Hence, applying the Poincaré inequality gives

$$
|b(w, w, u)| \leq c_3 \|w\|_1^{(2-\theta)} \|w\|_{L^2}^{\theta} \|u\|_1.
\tag{3.9}
$$

We define the Dirichlet quotient χ for solutions w of the equation (3.5)

$$
\chi = \frac{\|w\|_1}{\|w\|}.
\tag{3.10}
$$

If we consider the Poincaré inequality (2.5), we obtain from the above lemma that

$$
\frac{1}{\chi} \text{ is finite for } \|w(t)\| \neq 0, \ \forall t \in [0, T]
\tag{3.11}
$$

and

$$
0 < \frac{1}{\chi} \leq \frac{1}{\lambda_1^{1/2}}, \ \forall w \in V_1.
\tag{3.12}
$$

It follows that

$$\frac{1}{\chi}\left|b\left(w,w,u\right)\right| \leq c_3\|w\|_1^{(1-\theta)}\|w\|_{L^2}^{1+\theta}\|u\|_1. \tag{3.13}$$

Moreover, for $\|w\left(t\right)\| \neq 0$, Lemma 3.2 guarantees the existence of a constant $\mu > 0$ such that

$$\mu = \min_{t\in[0,T]}\left(\frac{1}{\chi}\right). \tag{3.14}$$

Using (3.13)-(3.14) in (3.5) and Young's inequality on the right-hand side, we obtain

$$\mu\frac{d}{dt}\|w\|^2 + \mu\nu\|w\|_1^2 \leq \frac{\nu\mu}{2}\|w\|_1^2 + c_4\left(\|w\|_{L^2}^{1+\theta}\|u\|_1\right)^{\frac{2}{1+\theta}}, \tag{3.15}$$

which gives

$$\mu\frac{d}{dt}\|w\|^2 + \frac{\nu\mu}{2}\|w\|_1^2 \leq c_4\|w\|^2\|u\|_1^{\frac{2}{\theta+1}}. \tag{3.16}$$

Dropping the positive term $\frac{\nu\mu}{2}\|w\|_1^2$, we get

$$\frac{d}{dt}\|w\|^2 \leq c_5\|u\|_1^{\frac{2}{\theta+1}}\|w\|^2. \tag{3.17}$$

Applying Gronwall's inequality on (3.17) and using the fact that

$$\frac{2}{\theta+1} \leq 2 \tag{3.18}$$

yields

$$\|w\left(t\right)\|^2 \leq c_6\|w\left(t_0\right)\|^2\int_{t_0}^t\|u\|_1^2 ds. \tag{3.19}$$

Since $u \in L^2(0,T;V_1)$, the integral expression on the right-hand side is finite, which implies both continuous dependence on initial conditions and uniqueness, this means that $w(x,t) = 0$ for all $t \geq t_0$ if $w\left(t_0\right) = 0$. □

Note that the terms in the right-hand side of (3.19) are independent of χ. Since the quantite $\frac{1}{\chi}$ is finite and definide for $\|w\left(t\right)\| \neq 0$. The same result holds for the 3D Navier-Stokes equations, provided Lemma 3.2 is replaced by a continuity assumption. More precisely, we have

Theorem 3.4. *Let $w = u - v$, the difference of two continuous weak solutions of the 3D Navier-Stokes equations, u and v, then we have $w\left(t_0\right) = 0$ implies $w\left(t\right) = 0$ for all $t \geq t_0$.*

This result gives a strong relation between the continuity and the uniqueness. Note that the continuity of the weak solutions of the 3-dimensional Navier-Stokes equations is known to be proved only in this weak sense [8, 11].

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